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Gradient-constrained minimum networks (II). Labelled or locally minimal Steiner points

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Abstract A *gradient-constrained minimum network* T is a minimum length network, spanning a given set of nodes N in space with edges whose gradients are all no more than an upper bound m. The nodes in T but not in N are referred to as *Steiner points*. Such networks occur in the underground mining industry where the typical maximal gradient is about 1:7 (≈ 0.14). Because of the gradient constraint the lengths of edges in T are measured by a special metric, called the gradient metric. An edge in T is labelled as a b-edge, or an m-edge, or an f-edge if the gradient between its endpoints is greater than, or equal to, or less than m respectively. The set of edge labels at a Steiner point is called its labelling. A Steiner point s with a given labelling is called *labelled minimal* if T cannot be shortened by a label-preserving perturbation of s. Furthermore, s is called *locally minimal* if T cannot be shortened by any perturbation of s even if its labelling is not preserved. In this paper we study the properties of labelled minimal Steiner points, as well as the necessary and sufficient conditions for Steiner points to be locally minimal. It is shown that, with the exception of one labelling, a labelled minimal Steiner point is necessarily unique with respect to its adjacent nodes, and that the locally minimal Steiner point is always unique, even though the gradient metric is not strictly convex.

Keywords Minimum networks · Gradient constrained · Steiner trees · Locally minimal

1 Introduction

A *gradient-constrained Steiner network* is a network interconnecting a given set of nodes N in 3-space, where each edge in the network is embedded so as to have gradient no greater than a given upper bound m. The given nodes in N are referred to as *terminals* while all other nodes (of degree 3 or more) of the network are referred to as *Steiner points* [9]. Such

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networks have many applications, particularly in the underground mining industry where the maximum gradient is generally about 1:7 (≈ 0.14) [5]. A *gradient-constrained minimum Steiner network* T is a gradient-constrained network whose total length is minimum for the given terminal set N. When searching for such a minimum network, only m and the positions of the terminals are fixed; the topology (underlying graph structure) of the network and positions of the edges and Steiner points are allowed to vary. In the context of underground mining, finding such a network corresponds to minimising the substantial infrastructure costs for the ramps and drives providing access to a given set of ore bodies [3,6].

Let x_p, y_p, z_p denote the Cartesian coordinates of a point p in Euclidean space. By the *gradient* of an edge pq we mean the absolute value of the slope from $p = (x_p, y_p, z_p)$ to $q = (x_q, y_q, z_q)$, which is denoted by $g(pq)$. That is,

$$g(pq) \overset{\text{def}}{=} \frac{|z_q - z_p|}{\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}}.$$

Using this notation, T can be regarded as a shortest network in a gradient metric space where the *gradient metric*, a combination of the Euclidean metric and the *vertical metric*, is defined as follows:

$$|pq|_g = \begin{cases} \sqrt{(x_p-x_q)^2+(y_p-y_q)^2+(z_p-z_q)^2} \text{ (Euclidean metric)} & \text{if } g(pq) \le m, \\ (\sqrt{1+m^{-2}})|z_p-z_q| \qquad \text{(vertical metric)} & \text{if } g(pq) \ge m. \end{cases} \tag{1}$$

(As usual, $|\cdot|$ is used to denote the Euclidean metric.) In gradient metric space every edge pq in T can be represented as a straight line segment. However, if the gradient of pq is greater than m, then it is straight in the gradient metric space but will be represented as a bent edge prq in Euclidean space, where pr, rq are straight line segments with maximum gradient m. We refer to r as a *corner point* on the bent edge. An edge pq is called an *f-edge, m-edge* or *b-edge* if pq is labelled 'f' ($g(pq) < m$), 'm' ($g(pq) = m$) or 'b' ($g(pq) > m$), respectively. Denote the horizontal plane through a Steiner point s by $\mathcal{H}_s$. For simplicity, a point or an edge will be said to be above (or below) s if it is above (or below) $\mathcal{H}_s$. A Steiner point is *degenerate* if it is coincident with an adjacent vertex.

Let a, b, c be three given vertices in a gradient-constrained network. Suppose a degree 3 (non-degenerate) Steiner point s adjacent to a, b, c has one incident edge as above s and the other two edges bs, cs below s. Let $\mathcal{L}_{as}, \mathcal{L}_{bs}, \mathcal{L}_{cs}$ denote the respective labels of these edges. Then we say the *labelling* of s is $(\mathcal{L}_{as}/\mathcal{L}_{bs}\mathcal{L}_{cs})$. Similarly, if as and bs are above s and cs is below s, then the labelling is $(\mathcal{L}_{as}\mathcal{L}_{bs}/\mathcal{L}_{cs})$. If s is of degree 4 with adjacent vertices a, b, c, d and as and ds lie above s while bs, cs are below s, then the labelling of s is denoted by $(\mathcal{L}_{as}\mathcal{L}_{ds}/\mathcal{L}_{bs}\mathcal{L}_{cs})$.

The focus of our study in paper [4] was on the labellings that can occur in a gradient-constrained minimum network T, and we proved the following fundamental facts:

Proposition 1.1 (1) *The degree of a Steiner point s in a gradient-constrained minimum network T is either three or four.*
(2) *Up to symmetry there are five* (*non-degenerate*) *feasibly optimal labellings* (*f/ff*), (*m/ff*), (*m/mf*), (*m/mm*) *and* (*b/mm*) *if s is of degree 3 and $m < 1$.*
(3) *If s is of degree 4 and if $m < 0.38$, then there is only one feasibly optimal labelling* (*mm/mm*). *Moreover, two edges, say as and bs, lie in a vertical plane, and so do the other two edges cs, ds.*

Constructing a gradient-constrained minimum Steiner network is a difficult global optimisation problem. An important step in the development of efficient methods for solving the global problem is to understand and and exploit properties of local minima. Indeed, this is a constant theme throughout geometric Steiner network theory. Hence, it is useful to identify a number of different types of minimality for the gradient-constrained problem, ranging from the most local to the global:

(1) *Labelled minimal*: minimal for a given topology and a given set of labellings on the edges. This means that no perturbation of a Steiner point that preserves the labelling of its incident edges can decrease the length of the network. Here we only need to consider labellings which induce feasibly optimal labellings at the Steiner points (i.e. labellings given in Proposition 1.1).
(2a) *Locally minimal* (*with respect to 1-point pertubations*): minimal for a given topology, with respect to any perturbation on a Steiner point (including perturbations that change edge labels).
(2b) *Locally minimal* (*with respect to all pertubations*): minimal for a given topology, with respect to all perturbations on Steiner points including those perturbations that simultaneously perturb more than one Steiner point (note that this is the usual definition of local minimality in the Euclidean Steiner problem, where (2a) and (2b) are equivalent).
(3) *Global minimum*: minimum for all possible topologies and all perturbations of Steiner points.

The aim of our previous paper [4] was to understand the properties of labelled minimal Steiner networks, and to determine the feasibly optimal labellings that need to be considered. In this present paper we investigate the relationship between labelled minimality and the local minimality of Case (2a). A future paper (in preparation) will study the local minimality of Case (2b) in terms of 1-point local minimality (Case (2a)).

By Proposition 1.1(3), if there exists a Steiner point s of degree 4 (and $m < 0.38$), then it is easily computed in terms of the coordinates of the adjacent vertices. Hence, for the remainder of this paper we ignore degree-4 Steiner points and only discuss degree-3 Steiner points. Moreover, we assume $m < 1$ so only the above five labellings in Proposition 1.1(2) are considered.

The definitions of labelled and local minimality above can also be applied to Steiner points. A Steiner point s with a given labelling is called *labelled minimal* if T cannot be shortened by label-preserving perturbations of s. Furthermore, s is called *locally minimal* if T cannot be shortened by any perturbation of s no matter whether its labelling is preserved or not. Although the gradient metric is not strictly convex, we will see that the labelled minimal Steiner point is unique. It follows that the locally minimal Steiner point is also unique since a locally minimal Steiner point must be labelled minimal. Clearly, a locally minimal Steiner point must be labelled minimal, and hence, can be selected from labelled minimal Steiner points. In this paper we study the properties of labelled minimal Steiner points, and the necessary and sufficient conditions for such points to be locally minimal. In particular, we show that some kinds of labelled minimal Steiner points appear to be algebraically unsolvable, i.e. cannot be determined by formulae with radicals.

2 Fundamentals

Throughout the rest of this paper s, with or without subscripts, denotes a degree 3 Steiner point with adjacent terminals a, b, c and T is the gradient-constrained tree consisting of the incident edges of s. Moreover, we assume $m < 1$ as required by Proposition 1.1(2). There

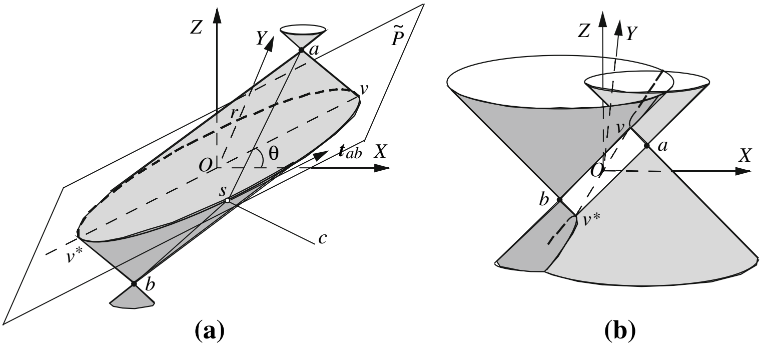

Fig. 1 Intersections of two cones

are two fundamental techniques required for studying the necessary and sufficient conditions for Steiner points to be labelled minimal or locally minimal.

2.1 Cone argument

Suppose p is a point. Let $\mathcal{C}_p$ denote the right circular cone whose vertex is p and the gradient of whose generating lines is m, the maximum permitted gradient. Such cones play a central role in determining Steiner points. Suppose a, b are distinct points, and $\mathcal{C}_a, \mathcal{C}_b$ are two such cones. If $g(ab) > m$ then $\mathcal{C}_a \cap \mathcal{C}_b$ is an ellipse [4]. In an unpublished research note [10] one of the authors has made an exhaustive study on conic sections produced by two right circular cones. Below are some results cited from the note and used in this paper. A sketch of the proof is given.

Theorem 2.1 *Suppose $\mathcal{C}_a, \mathcal{C}_b$ are two right circular cones whose generating lines have the same gradient m. Without loss of generality assume $a = (u, 0, h),\ b = (-u, 0, -h),\ u \geq 0, h \geq 0$.*

(1) *The intersection $\mathcal{E}_{ab}$ (or simply $\mathcal{E}$) of $\mathcal{C}_a, \mathcal{C}_b$ is either an ellipse or a hyperbola depending on whether $g(ab) > m$ or $g(ab) < m$, respectively. When $g(ab) = m$, $\mathcal{E}_{ab}$ degenerates into a straight line through a, b.*
(2) *The ellipse or hyperbola has two extreme points $v = (h/m, 0, mu)$, $v^* = (-h/m, 0, -mu)$, referred to as its vertices.*
(3) *Let $s \in \mathcal{E}$. If $\mathcal{E}$ is an ellipse then $|as| + |bs| = 2z_a/\sin\alpha$ is constant, where $\alpha = \arctan(m)$. If $\mathcal{E}$ is a hyperbola then $|as| + |bs|$ achieves its minimum when $s = v$ (or v^*).*
(4) *For any point $s \in \mathcal{E}$ let $\mathbf{t}_{ab}$ be the (upward) tangent vector at s. Then the gradient of $\mathbf{t}_{ab}$ is less than m. Moreover, if $\mathcal{E}$ is an ellipse, then $\angle(\overrightarrow{sa}, \mathbf{t}_{ab}) \leq 90°$, $\angle(\overrightarrow{sb}, -\mathbf{t}_{ab}) \leq 90°$, and equality holds if and only if $s = v$ or $s = v^*$. If $\mathcal{E}$ is a hyperbola and s lies on the upper branch, then $\angle(\overrightarrow{sa}, -\mathbf{t}_{ab}) \leq 90°$, $\angle(\overrightarrow{sb}, -\mathbf{t}_{ab}) \leq 90°$, and the equality holds if and only if $s = v$. The results are similar if s lies on the lower branch.*

Proof (Sketch) Let $s = (x_s, y_s, z_s)$ be a point on the intersection of the right circular cones $\mathcal{C}_a$ and $\mathcal{C}_b$ (Fig. 1). Then the two cones are determined by

$$(x_s - u)^2 + y_s^2 = \frac{(z_s - h)^2}{m^2}, \quad (x_s + u)^2 + y_s^2 = \frac{(z_s + h)^2}{m^2}. \tag{2}$$

Subtracting the first equation from the second, we obtain the equation for the intersection $\mathcal{E}$:

$$\frac{z_s}{x_s} = m^2 \frac{u}{h}, \quad \text{i.e. } z_s h = m^2 u x_s. \tag{3}$$

This equation implies that $\mathcal{E}$ lies on a plane $\tilde{P}$, which is through the y-axis and between a and b. Hence, $\mathcal{E}$ is an ellipse or a hyperbola. By standard calculus operations it is not hard to prove the other statements in the theorem. □

Remark 2.1 In general, if $g(as) = m$ we define $\mathcal{P}(as)$ to be the *tangent plane* containing as and tangent to $\mathcal{C}_a$. Similarly, we define $\mathcal{P}(bs)$. Then, in Theorem 2.1 the intersection $\mathcal{P}(as) \cap \mathcal{P}(bs)$ of the two tangent planes determines the vector $\mathbf{t}_{ab}$.

2.2 Variational argument in gradient metric space

The variational method [7] has proved to be a powerful tool in the study of the Euclidean Steiner tree problem. We can generalise it to the gradient-constrained Steiner tree problem. Note that the gradient metric defined by Formula (1) is a combination of the Euclidean metric and a special metric (referred to as the *vertical metric*) for edges whose gradients are $\leq m$ and $\geq m$, respectively. Note also that the vertical metric is convex but not strictly convex and, consequently, so is the gradient metric. Suppose $e = ps$ is an edge in gradient metric space. Let $L_e = |pq|_g$ be the length of e, and $L_T = \sum_{e \in T} L_e$ be the tree length of T. Then it is easy to check that the following lemmas hold from the definition of the gradient metric:

Lemma 2.2 [4] *Suppose $e = ps$ is an edge and s is perturbed to s' in direction $\mathbf{v}$ with p being fixed. Let $\dot{L}_e(\mathbf{v})_g$ be the directional derivative of L_e in gradient metric space.*

(1) *If $g(ps) < m$ then $\dot{L}_e(\mathbf{v})_g = -\cos \angle pss'$.*
(2) *If $g(ps) > m$ then $\dot{L}_e(\mathbf{v})_g = -\sqrt{1 + m^{-2}} \cos \angle zss'$ where z is a point on the vertical line through s such that $\angle psz \leq 90°$,*
(3) *If $g(ps) = m$ then $\dot{L}_e(\mathbf{v})_g = -\cos \angle pss'$ or $= -\sqrt{1 + m^{-2}} \cos \angle zss'$ depending on whether $g(ps') \leq m$ or $g(ps') \geq m$, respectively.*

Remark 2.2 From Lemma (2.2) we can see that, as in Euclidean space [7], the variation of an edge in the gradient metric space is determined only by the angle at which one perturbs its endpoints, and is independent of the edge's length.

Lemma 2.3 *Suppose two straight edges as, bs in T meet at an angle less than 90°. T is not minimal if as is an f-edge, or if as is an m-edge and lies on the same side of $\mathcal{H}_s$ as bs.*

Proof Let s be perturbed along sb to s'. Then the lemma is trivial if as is an f-edge or if as is an m-edge and as' is an f-edge. If as is an m-edge and as' is an m-edge or b-edge then $\dot{L}_{as} < 0$ since the vertical distance between the endpoints of as is decreasing. □

We will omit the subscript g in $\dot{L}_e(\mathbf{v})_g$ or $\dot{L}_T(\mathbf{v})_g$ if no confusion is caused. For constructing a coordinate system in gradient metric space, it is required to choose three non-coplanar vectors as axes. Our usual strategy in later parts of this paper will be to define the new axes according to the labelling of s based on the following set of principles:

- The origin is set at s.
- For each pair of edges incident with s, say as and bs we choose a corresponding axis as follows:

  a If as and bs are both m-edges we choose an axis along the line $\mathcal{P}(as) \cap \mathcal{P}(bs)$;

b If only one edge, say as, is an m-edge we choose an arbitrary axis lying on the tangent plane $\mathcal{P}(as)$;
c If neither edge is an m-edge then the corresponding axis is chosen arbitrarily.

- The above choices are made in such a way that the three axes are not co-planar.

Clearly such axes can always be constructed, unless we have three m-edges lying in a vertical plane. The unit vectors in the positive directions of the three axes will be denoted by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, respectively. Note that by such a choice of axes, the gradient metric space is partitioned into eight octants, where the tangent planes of the m-edges are involved in this partitioning, This implies that all perturbations of one of the incident edges into a single octant involve only one metric, either the Euclidean or the vertical metric. Hence, the following theorem and corollary hold.

Theorem 2.4 *Suppose $e = ps$ with p fixed and s being perturbed in direction $\mathbf{v}$. Suppose the axes are chosen as stated above. Let $\mathbf{v}$ be decomposed as $\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z$ where $\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z$ are the projections of $\mathbf{v}$ on the x-axis, y–axis and z-axis respectively. Then*

$$\mathrm{d}L_e(\mathbf{v}) = \dot{L}_e(\mathbf{v}_x)\mathrm{d}x + \dot{L}_e(\mathbf{v}_y)\mathrm{d}y + \dot{L}_e(\mathbf{v}_z)\mathrm{d}z. \tag{4}$$

Corollary 2.5 *The point s, in the previous theorem, is locally minimal if and only if $\dot{L}_T(\pm\mathbf{x}) \geq 0$, $\dot{L}_T(\pm\mathbf{y}) \geq 0$, $\dot{L}_T(\pm\mathbf{z}) \geq 0$.*

A perturbation of s with respect to e (or to T) is called *reversible* if $\dot{L}_e(\mathbf{v}) = -\dot{L}_e(-\mathbf{v})$ [or $\dot{L}_T(\mathbf{v}) = -\dot{L}_T(-\mathbf{v})$, respectively]. The gradient metric is not differentiable when the gradient is m, consequently not all perturbations are reversible. The following theorem trivially holds.

Theorem 2.6 *Suppose $e = ps$ with p fixed and s being a Steiner point. If e is an f-edge or a b-edge, then the label of e is preserved under any perturbation of s, and the perturbation is reversible with respect to e. However, if e is an m-edge and if $\mathbf{v}$ does not lie in $\mathcal{P}(ps)$ then the perturbation is not reversible.*

It follows from this theorem that if s has incident m-edges then to show s is locally minimal we need to consider the perturbations of s in both positive and negative directions of $\mathbf{v}$.

3 Labelled minimal Steiner points

From now on we assume that T has a degree 3 (non-degenerate) labelled minimal Steiner point s with one edge as above s and two edges bs, cs below s. We assume that s has one of the five feasibly optimal labellings listed in Proposition 1.1(2). The geometric properties of labelled minimal s were briefly discussed in Ref. [4]. The objectives of this paper are to study the following two problems:

(P1) *Find the simplest system of equations that determines s.*
Ideally, the system should consist of 3 independent equations. If s has k $(0 \leq k \leq 3)$ incident m-edges, then the gradient constraints of the k m-edges form part of the system. The remaining $(3 - k)$ equations in the system can be determined by the variational argument. Where possible, we will also determine the degree of the system and identify the minimum solution if the system has more than one real solution.

(P2) *Decide whether the labelled minimal Steiner point s is also locally minimal, and, if not, find the direction in which to perturb this node so as to reduce the tree length.*
In this study we assume that the labelled minimal s satisfies Lemma 2.3, otherwise s cannot be locally minimal since we can simply move s as described in the lemma to shorten T. Moreover, if s does not have an incident m-edge, then s being labelled minimal implies s is also locally minimal by Theorem 2.6. On the other hand, by Corollary 2.5 if there are m-edges then we need consider both positive and negative directions for perturbations along these edges. Later we will see that if s has incident m-edges and is labelled minimal but not locally minimal, then any length-reducing perturbation of s results in a change of labelling that transforms at least one m-edge into an f-edge or a b-edge.

We study the two problems labelling by labelling. Our main focus is on labellings (m/mm) as case is the most complicated.

3.1 Labelling (b/mm)

As discussed at the beginning of this section, two of the equations for determining s are the gradient constraints $g(sb) = g(sc) = m$. That is, s lies on the upper branch of the hyperbola $\mathcal{E}_{bc} = \mathcal{C}_b \cap \mathcal{C}_c$. By Lemma (2.1)(3) s must be the vertex v of the hyperbola. It follows that s lies in the vertical plane through bc and the projections of b, s, c on a horizontal plane lie in a straight line. Thus, the system determining s is

$$g(sb) = g(sc) = m, \quad \frac{y_s - y_c}{x_s - x_c} = \frac{y_b - y_c}{x_b - x_c}. \tag{5}$$

Note that as shown in Formula (3) after eliminating a variable, the 2 equations $g(sb) = g(sc) = m$ become a linear equation. Therefore, the system (5) is linear and there is only one solution. The uniqueness is trivial from the geometric point of view: the vertex of a hyperbola (in a branch) is unique.

Theorem 3.1 (1) *The labelled minimal point s for the labelling* (*b/mm*) *is the vertex v of $\mathcal{E}_{bc}$, which is uniquely determined by the linear system of Equation (5).*
(2) *s is locally minimal as well as labelled minimal.*

Proof The first statement has been proved above. To prove the second statement we re-set the axes such that s is the origin and that $\mathbf{y} = -\overrightarrow{sb}$ and $\mathbf{z} = -\overrightarrow{sc}$ are the y-axis and z-axis, respectively. The intersection of the tangent planes $\mathcal{P}(bs)$ and $\mathcal{P}(cs)$ is a horizontal line and is taken as the x-axis (its positive direction can be chosen arbitrarily). The observations below follow from the definition of the gradient metric:

a $\dot{L}_{as}(\pm\mathbf{x}) = 0$ and $\dot{L}_{bs+cs}(\pm\mathbf{x}) > 0$. Hence $\dot{L}_T(\pm\mathbf{x}) > 0$.
b $\dot{L}_{as+cs}(\mathbf{y}) = 0$ and $\dot{L}_{bs}(\mathbf{y}) > 0$. Hence $\dot{L}_T(\mathbf{y}) > 0$.
c $\dot{L}_{as+bs}(-\mathbf{y}) = 0$ and, since $m < 1$ we have $\angle bsc > 90°$, so $\dot{L}_{cs}(-\mathbf{y}) > 0$. Hence $\dot{L}_T(-\mathbf{y}) > 0$.

By symmetry, we also have $\dot{L}_T(\pm\mathbf{z}) > 0$. Thus by Corollary 2.5 the second statement holds. □

3.2 Labelling (m/mm)

Clearly, s is determined by the system of equations

$$g(as) = g(bs) = g(cs) = m. \tag{6}$$

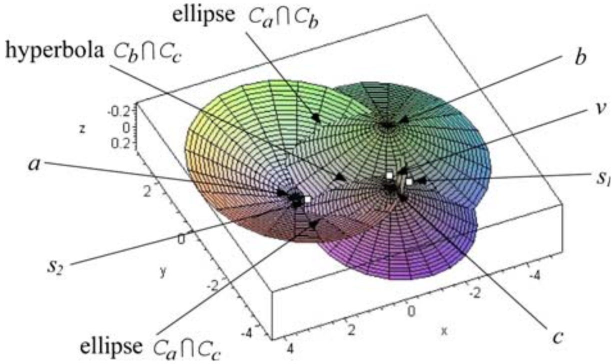

Fig. 2 Intersection of three cones

It follows that s is an intersection of three cones, and hence an intersection of three conics: two ellipses and one hyperbola. Figure 2 is a 3D-plot of the intersection of the three cones [to make the picture clear, the plot is for the labelling (mm/m), not for (m/mm)]. Note that $g(b, c) < m$, since otherwise s collapses into b or c, contradicting the assumption that s is non-degenerate. Note also that $g(a, b) \geq m$ and $g(a, c) \geq m$ due to the labelling. Since the hyperbola $\mathcal{E}_{bc} = \mathcal{C}_b \cap \mathcal{C}_c$ and the cone $\mathcal{C}_a$ are both convex, they meet at no more than two points, that is, the above system of equations has at most two real solutions.

The analytical argument is as follows. As in Case (b/mm), after eliminating a variable, $g(bs) = g(cs) = m$ becomes a linear equation. Combining these linear equations with the equation $g(as) = m$ then results in a quadratic equation. Hence the system (6) is of degree 2 and has at most two real different solutions.

Theorem 3.2 *The labelled minimal Steiner point s for labelling (m/mm) is determined by the quadratic system of eq. 6, which has at most two real different solutions s_1, s_2, lying on the hyperbola $\mathcal{E}_{bc}$. If $z_{s_1} = z_{s_2}$, then both points are labelled minimal, otherwise the one with the smaller z-coordinate is the labelled minimal Steiner point.*

Proof The first part has been proved above. Let T_1, T_2 be the trees with Steiner points s_1, s_2, respectively. Because

$$|as_1|_g + |bs_1|_g = |as_2|_g + |bs_2|_g \text{ and } |as_1|_g + |cs_1|_g = |as_2|_g + |cs_2|_g,$$

$$2(|T_1|_g - |T_2|_g) = (|bs_1|_g + |cs_1|_g) - (|as_2|_g + |cs_2|_g) \leq 0 \text{ if and only if } z_{s_1} \leq z_{s_2}.$$

This proves the second part of the theorem. □

Theorem 3.3 *Let s be the labelled minimal Steiner point for (m/mm). Let v be the vertex of the hyperbola $\mathcal{E}_{bc}$.*

(1) *If v lies strictly inside $\mathcal{C}_a$, then s is not locally minimal.*
(2) *If v lies on the surface of $\mathcal{C}_a$ (i.e. $s = v$), then s is locally minimal.*
(3) *If v lies strictly outside $\mathcal{C}_a$, then s may or may not be locally minimal, depending on the position of s.*

The point s is necessarily unique except in the Case (1), in which however the resulting locally minimal Steiner point is unique.

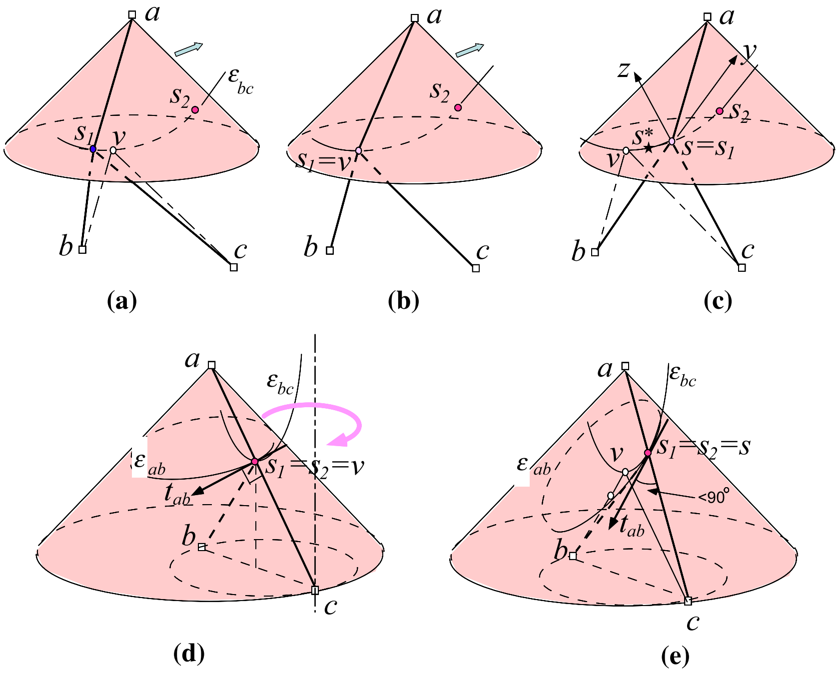

Fig. 3 Labelled Steiner points for labelling m/mm

Proof As $g(ab) \geq m$ and $g(ac) \geq m$, either both inequalities are strict or exactly one of them is an equality. Using this criterion, Cases (2) and (3) discussed below will be further divided into subcases, referred to as type A and type B, respectively.

(1) If v lies strictly inside $\mathcal{C}_a$ then $g(ab) > m$, $g(ac) > m$, and (using the notation of Theorem 3.2) $s_1 \neq s_2$. This is illustrated in Fig. 3a. Let $T_{\mathbf{v}}$ be the tree with $\mathbf{v}$ as its Steiner point. Then, as in the proof of Theorem 3.2, we have $L_{T_{\mathbf{v}}} < L_{T_1}$, $L_{T_{\mathbf{v}}} < L_{T_2}$. Hence, at least one of s_1, s_2 is labelled minimal but neither point is locally minimal. In this case v is the locally minimal Steiner point for T and has labelling (b/mm).

(2A) If $s = v$ lies on the surface of $\mathcal{C}_a$ and if $g(ab) > m$, $g(ac) > m$, then $s = s_1 = v$ and $s_1 \neq s_2$ as shown in Fig. 3b. An m-edge is a critical b-edge and therefore the labelling (m/mm) is a critical case of the labelling (b/mm). By similar arguments to those in the proof of Theorem 3.2 it follows that T_1 cannot be shortened by moving s in any direction, and s is locally minimal.

(2B) Suppose $s = v$ lies on the surface of $\mathcal{C}_a$ but one of $g(ab)$ and $g(ac)$, say $g(ac)$, equals m. In this case, the ellipse $\mathcal{E}_{ac}$ degenerates to the line through a and c and $s_1 = s_2 = v$ lies on the line segment ac (Fig. 3d). Note that v lying on the surface of $\mathcal{C}_a$ forces b to lie in the vertical plane through ac, and note that v is also the vertex of the ellipse $\mathcal{E}_{ab}$. Hence, as proved in Ref. [2], $s = v$ is locally minimal as well as labelled minimal.

(3A) If v lies outside $\mathcal{C}_a$ and if $g(ab) > m$, $g(ac) > m$, then $s_1 \neq s_2$ and both lie on the upper branch of the hyperbola $\mathcal{E}_{bc}$ on one side of v as shown in Fig. 3c. In this case $s = s_1$ is labelled minimal but is not necessarily locally minimal, depending on the position of s.

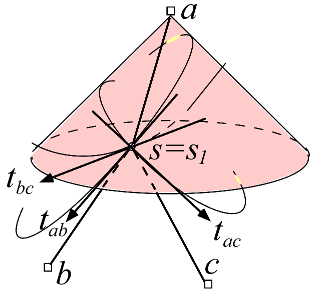

Fig. 4 Perturbation of s along tangents

(3B) Suppose v lies strictly outside $\mathcal{C}_a$ and either $g(ab)$ or $g(ac)$, say $g(ac)$, is m. As in Case (2B) the ellipse $\mathcal{E}_{ac}$ degenerates to a line through a and c and the upper branch of the hyperbola $\mathcal{E}_{bc}$ just touches $\mathcal{C}_a$ at a point on ac, which is the labelled minimal point $s = s_1 = s_2$(Fig. 3e). Let s move downwards along the tangent $\mathbf{t}_{ab}$ of the ellipse $\mathcal{E}_{ab}$ as shown in the figure. Then $\angle(\overrightarrow{sb}, \mathbf{t}_{ab}) < 90°$ by Theorem 2.1. Hence, $\dot{L}_{as+sc}(\mathbf{t}_{ab}) = 0$, $\dot{L}_{bs}(\mathbf{t}_{ab}) = -\cos\angle(\overrightarrow{sb}, \mathbf{t}_{ab}) < 0$. It follows that $\dot{L}_T < 0$ and s is not locally minimal.

□

Remark 3.1 In order to visualise the different cases in Fig. 3 it is helpful to consider what happens as a changes position relative to b and c. Beginning with $g(ab) > m$ and $g(ac) > m$, if a (and hence cone $\mathcal{C}_a$) moves horizontally away from bc, then Fig. 3a changes into Fig. 3b and then into Fig. 3c. When $g(ac) = m$, if a, together with cone $\mathcal{C}_a$, rotates around a vertical line through c, then Fig. 3d changes into Fig. 3e.

In Case (3A) of the above theorem, let $\mathbf{x} = \mathbf{t}_{ab}$ be the downward tangent vector of the ellipse $\mathcal{E}_{ab}$ at s. Similarly, define the downward tangent vector $\mathbf{y} = \mathbf{t}_{ac}$ of the ellipse $\mathcal{E}_{ac}$ and the downward tangent vector $\mathbf{z} = \mathbf{t}_{bc}$ of the hyperbola $\mathcal{E}_{bc}$ (Fig. 4). The 3 tangent vectors are non-coplanar, and define a set of axes which allow us to apply Theorem 2.4.

Theorem 3.4 *Let axes be given by the above downward tangent vectors. Then $s = s_1$ is not locally minimal if and only if at least one of $\dot{L}_T(\mathbf{x}) < 0$, $\dot{L}_T(\mathbf{y}) < 0$ and $\dot{L}_T(\mathbf{z}) < 0$ holds.*

Proof By Theorem 2.4 we only need to prove that the variation is positive when s moves in the negative axis directions. First note that $\mathbf{x}$ is not horizontal. When s moves in the upward direction $-\mathbf{x}$, $|as| + |bs|$ is constant but $|cs|$ increases. Hence $\dot{L}_T(-\mathbf{x}) > 0$. Similarly $\dot{L}_T(-\mathbf{y}) > 0$. If s moves in the upward direction $-\mathbf{z}$, then both arguments hold: $|as| + |bs|$ is constant with $|cs|$ increasing and $|as| + |cs|$ is constant with $|bs|$ increasing. Hence, $\dot{L}_T(-\mathbf{z}) > 0$. □

Let s be perturbed to s' along one of the axes such that $|ss'|$ is arbitrarily small. The above discussion is summarized in Table 1 in which "≥ 0?" indicates that that the inequality may or may not hold (depending on the position of s).

The labelling change for the perturbation in direction $\mathbf{x}$ is particularly interesting: the resulting labelling (f/mm) is an infeasible labelling. In that case s moves downwards from s' continuing to reduce the tree length until s once again reaches the surface of $\mathcal{C}_a$. As a result the possible labelling for local minimality is (m/ff).

Table 1 Perturbations of s with labelling (m/mm)

	$\mathbf{x}$	$\mathbf{y}$	$\mathbf{z}$	$-\mathbf{x}$	$-\mathbf{y}$	$-\mathbf{z}$
$\mathcal{L}_{as'}$	m	m	f	m	m	b
$\mathcal{L}_{bs'}$	m	f	m	m	b	m
$\mathcal{L}_{cs'}$	f	m	m	b	m	m
$\dot{L}_T$	≥ 0?	≥ 0?	≥ 0?	>0	>0	>0

Theorem 3.5 *The Steiner point s is locally minimal if and only if $\dot{L}_T(\overrightarrow{as}) \geq 0$.*

Proof Note that all three angles $\angle(\overrightarrow{as}, \mathbf{t}_{ab})$, $\angle(\overrightarrow{as}, \mathbf{t}_{ac})\angle(\overrightarrow{as}, \mathbf{t}_{bc})$ are less than 90°. Hence $\overrightarrow{as}$, lying in the first octant, can be decomposed into a sum of its projections on the three tangent vectors as argued in Theorem 2.4. Therefore,

$$\begin{aligned} s \text{ is locally minimal } &\Leftrightarrow \neg\big((\dot{L}_T(\mathbf{x}) < 0) \cup (\dot{L}_T(\mathbf{y}) < 0) \cup (\dot{L}_T(\mathbf{z}) < 0)\big) \\ &\Leftrightarrow (\dot{L}_T(\mathbf{x}) \geq 0) \cap (\dot{L}_T(\mathbf{y}) \geq 0) \cap (\dot{L}_T(\mathbf{z}) \geq 0) \\ &\Leftrightarrow \dot{L}_T(\overrightarrow{as}) \geq 0. \end{aligned}$$

□

3.3 Labelling (m/mf)

To determine s, first we have two equations $g(as) = g(sb) = m$. It follows $g(ab) \geq m$. By the variational argument the third condition for determining s is that cs should be perpendicular to the ellipse $\mathcal{E} = \mathcal{E}_{ab}$ (or its degeneration ab) at s (see Fig. 1a). Let $\mathbf{t}_{ab}$ be the tangent vector at s. Then, the equations determining s are

$$g(as) = g(sb) = m, \quad \angle(\overrightarrow{sc}, \mathbf{t}_{ab}) = 90°. \tag{7}$$

We claim that this system is of degree 4. The problem of finding s is equivalent to the problem of finding a normal line from a point in space to an ellipse on a plane. It has been shown that this problem requires solving a degree 4 polynomial [1]. Furthermore, we claim that the system has only one real solution. The reason is that the ellipse $\mathcal{E}$ is strictly convex and, by the labelling, c must be outside $\mathcal{C}_a$ and $\mathcal{C}_b$. Therefore, the labelled minimal Steiner point s is unique and the following theorem holds.

Theorem 3.6 *The labelled minimal Steiner point s for labelling (m/mf) is determined by the quartic system of Equation (7), which has only one real solution.*

Theorem 3.7 *Let s be the labelled minimal Steiner point for (m/mf).*

(1) *If $g(ab) = m$, then s is not locally minimal.*
(2) *If $g(ab) > m$, then s is locally minimal if and only if $\dot{L}_T(\mathbf{x}) \geq 0$ where $\mathbf{x} = -\overrightarrow{sa}$.*

Proof (1) If $g(ab) = m$, then s lies on ab and $sc \perp ab$. If s is perturbed in direction $\overrightarrow{sc}$ then $\dot{L}_{as+bs}(\overrightarrow{sc}) = 0$ and $\dot{L}_{sc}(\overrightarrow{sc}) < 0$. Hence, $\dot{L}_T(\overrightarrow{sc}) < 0$ and s is not locally minimal.

(2) Let $\mathbf{y} = -\overrightarrow{sb}$, $\mathbf{z} = \mathbf{t}_{ab}$ and apply Corollary 2.5. It is easy to verify that $\dot{L}_T(-\mathbf{x}) > 0$ and $\dot{L}_T(-\mathbf{y}) > 0$. Since $\mathbf{t}_{ab}$ is tangent to $\mathcal{E}_{ab}$ and the ellipse $\mathcal{E}_{ab}$ is convex, it follows that

Table 2 Perturbations of s with labelling (m/mf)

	$\mathbf{x}$	$\mathbf{y}$	$-\mathbf{x}$	$-\mathbf{y}$	$\pm\mathbf{z}$
$\mathcal{L}_{as'}$	m	f	m	b	f
$\mathcal{L}_{bs'}$	f	m	b	m	f
$\mathcal{L}_{cs'}$	f	f	f	f	f
$\dot{L}_T$	≥0?	≥0*	>0	>0	≥0

$\dot{L}_T(\pm\mathbf{z}) = 0$ and $\ddot{L}_T(\pm\mathbf{z}) > 0$. Hence, T cannot be shortened by moving s in the tangent direction $\mathbf{z}$. Now note that $\angle(\mathbf{y}, \overrightarrow{sc}) > \angle(\mathbf{x}, \overrightarrow{sc})$ implies

$$\begin{aligned}\dot{L}_T(\mathbf{y}) &= 1 - \cos\angle(\mathbf{y}, \overrightarrow{sa}) - \cos\angle(\mathbf{y}, \overrightarrow{sc})\\ &= 1 - \cos\angle(\mathbf{x}, \overrightarrow{sb}) - \cos\angle(\mathbf{y}, \overrightarrow{sc})\\ &> 1 - \cos\angle(\mathbf{x}, \overrightarrow{sb}) - \cos\angle(\mathbf{x}, \overrightarrow{sc}) = \dot{L}_T(\mathbf{x}).\end{aligned}$$

Hence, $\dot{L}_T(\mathbf{x}) > 0$ implies $\dot{L}_T(\mathbf{y}) > 0$ and the statement is proved. □

Let s be perturbed to s' such that $|ss'|$ is arbitrarily small. The above discussion is summarized in Table 2 in which "$\dot{L}_T(\mathbf{x}) \le 0$?" indicates that the inequality may or may not hold and "$\dot{L}_T(\mathbf{y}) \ge 0$*" indicates that it holds under the condition $\dot{L}_T(\mathbf{x}) \ge 0$.

3.4 Labelling (m/ff)

Since s, as a labelled minimal Steiner point, is constrained on cone $\mathcal{C}_a$, the first equation determining s is $g(as) = m$. Let $\mathbf{x} = -\overrightarrow{sa}$ and let $\mathbf{y}$ be the vector tangent to the circle $\mathcal{C}_a \cap \mathcal{H}_s$, which is perpendicular to $\mathbf{x}$. By the variational argument, s is determined by the following equations (from Ref. [4, Theorem 7]:

$$\begin{aligned}f_x &= g(sa) = m,\\ f_t &= \cos\angle(\overrightarrow{sb}, \mathbf{y}) + \cos\angle(\overrightarrow{sc}, \mathbf{y}) = 0,\\ f_n &= 1 - \cos\angle(\overrightarrow{sb}, \mathbf{x}) - \cos\angle(\overrightarrow{sc}, \mathbf{x}) = 0.\end{aligned} \tag{8}$$

In terms of vectors, f_t, f_n can be re-written as

$$f_t = \frac{\overrightarrow{sb}\cdot\mathbf{y}}{|sb|} + \frac{\overrightarrow{sc}\cdot\mathbf{y}}{|sc|} = 0,$$

$$f_n = \frac{\overrightarrow{sb}\cdot\mathbf{x}}{|sb|\,|\mathbf{x}|} + \frac{\overrightarrow{sc}\cdot\mathbf{x}}{|sc|\,|\mathbf{x}|} - 1 = 0.$$

Because $\mathcal{C}_a$ is a convex surface, there is only one labelled minimal Steiner point s if it exists. After eliminating s_z using f_x and removing radicals by squaring, f_t is transformed into a degree 4 polynomial in s_x, s_y, and f_n a degree 8 polynomial. Hence, we have the following theorem.

Theorem 3.8 *The labelled minimal Steiner point s for labelling (m/ff) is determined by the degree 8 system of eq. 8, which has a unique real solution.*

Table 3 Perturbations of s with labelling (m/ff)

	$\pm\mathbf{x}$	$\pm\mathbf{y}$	$-\mathbf{z}$	$\mathbf{z}$
$\mathcal{L}_{as'}$	m	m	b	f
$\mathcal{L}_{bs'}$	f	f	f	f
$\mathcal{L}_{cs'}$	f	f	f	f
$\dot{L}_T$	$=0$	$=0$	>0	≥ 0?

Table 4 Labelled minimal Steiner points

Labelling	Equations	Degree	Number
(b/mm)	Equations (5)	linear	1
(m/mm)	Equations (6)	quadratic	2
(m/mf)	Equations (7)	quartic	1
(m/ff)	Equations (8)	degree 8	1
(f/ff)	Equations (9)	quadratic	1

We would expect this system of equations to generally be unsolvable by radicals. Our analysis of examples of such labelled minimal Steiner points appears to confirm this conjecture.

Let $-\mathbf{z}$ be the projection of $\overrightarrow{sa}$ onto the horizontal plane $\mathcal{H}_s$.

Theorem 3.9 *The labelled minimal Steiner point s for (m/ff) is locally minimal if and only if $\dot{L}_T(\mathbf{z}) \geq 0$.*

Proof Let the three non-coplanar vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be chosen as above. Since s is labelled minimal, then, by a similar argument to that in Theorem 3.7(2) we have $\dot{L}_T(\pm\mathbf{x}) = 0$, $\ddot{L}_T(\pm\mathbf{x}) > 0$, $\dot{L}_T(\pm\mathbf{y}) = 0$, $\ddot{L}_T(\pm\mathbf{y}) > 0$. That is, s is minimised with respect to perturbations in $\mathbf{x}$, $\mathbf{y}$. If s is perturbed in $-\mathbf{z}$, then $\dot{L}_{as}(-\mathbf{z}) = 0$ and $\dot{L}_{bs+cs}(-\mathbf{z}) > 0$. Thus, $\dot{L}_T(-\mathbf{z}) > 0$ and by Corollary 2.5 the theorem holds □

It is easily checked that if, alternatively, we choose $\mathbf{z}$ to be $\mathbf{n}_{\mathcal{P}(as)}$, the outward unit normal vector to $\mathcal{C}_a$ at s, then Theorem 3.9 still applies. For either definition of $\mathbf{z}$, the effect of perturbing s along one of the axes to s' is summarized in Table 3 in which "$\leq$0?" indicates that the inequality may or may not hold.

Another useful geometric condition for local minimality is the following.

Theorem 3.10 *Let $\mathcal{P}$ be the plane containing $\triangle abc$, and suppose $\mathcal{P}$ intersects $\mathcal{C}_a$ only at a. Then s is not locally minimal.*

Proof If s is perturbed towards its projection onto $\mathcal{P}$ then all edges incident with s become f-edges. Hence the pertubation behaves like a Euclidean perturbation and reduces $|T|$. □

3.5 Labelling (f/ff)

Since the label of an f-edge is preserved under any perturbation (Theorem 2.6), in this case the vertical metric is not active. Hence, the problem becomes a Euclidean Steiner problem and the following theorem holds [8].

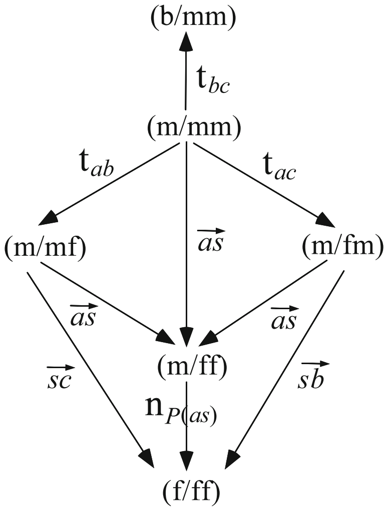

Fig. 5 Changes of labellings if the labelled minimal s is not locally minimal

Theorem 3.11 *The labelled minimal Steiner point s for labelling* (*f/ff*) *is determined by*

$$\angle asb = \angle bsc = \angle csa = 120^\circ. \tag{9}$$

This system of equations is quadratic and has a unique real solution s. The Steiner point s is locally minimal as well as labelled minimal.

4 Concluding remarks

1. The following table summarizes the answer to Problem (P1) raised at the beginning of the last section. The table shows the system of equations determining the labelled minimal Steiner point s, the degree of the system, and the maximum number of real solutions.
2. If the labelled minimal Steiner point s is not locally minimal, then the length of T can be reduced by perturbing s in a certain direction $\mathbf{v}$ and changing the labelling of s. For each such case, Fig. 5 shows a suitable direction $\mathbf{v}$ and the resulting change in labelling.

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